

# Measurable genuine tripartite entanglement of $(2 \otimes 2 \otimes n)$ -dimensional quantum states via only two simultaneous copies

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Usually, the three-tangle of a tripartite pure state of qubits can be directly measured with the simultaneous preparation of a not-less-than-four-fold copy of the state. We show that the exact genuine tripartite entanglement for  $(2 \otimes 2 \otimes n)$ -dimensional pure quantum states can be measured in a similar manner, provided that only two simultaneous copies of the state are available. Lower bounds are also proposed for more convenient experimental operations. As an example, a comprehensive demonstration of the scheme is provided for the three-tangle of a three-qubit state.

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## I. INTRODUCTION

Quantum entanglement is the combination of quantum superposition and the tensor product structure of quantum state space. It is one of the most fundamental features of quantum mechanics which distinguishes the quantum from classical world, while it serves as an important physical resource in quantum information processing tasks. In past decades, quantification of entanglement, one of the key subjects in entanglement theory, has attracted much interest and a lot of remarkable entanglement measures have been proposed [1]. However, quantum entanglement, in general, does not correspond to an observable due to the unphysical operations such as the complex conjugation for concurrence [2] and the partial transpose for negativity [3, 4]. This means that entanglement can not be directly measured in experiment. So the usual method to measuring entanglement is reconstructing the density matrix to be considered by the state tomography [5–7] which is fit for small systems. As an effective tool of detecting entanglement, the entanglement witness [8] usually depends on the state, which implies some prior knowledge about the state. Even though some methods have been developed in order to overcome the mentioned shortcomings [9–13], an important step is taken by the reformulation of pure-state concurrence of qubits in terms of a series of projectors on the two-fold copy of the state [14, 15]. This is also a direct motivation for both the latter experimental realization [16, 17] and the theoretic progress on the measurable concurrence for mixed states [18–20], measurable geometric discord of two qubits [21, 22], and measurable three-tangle of pure states [23]. Of course, the two-fold (or multiple-fold) copy should be understood as a source producing identical states to the one for which we want to quantify its entanglement, which should be distinguished from the quantum non-cloning theorem [24]. Although a not-less-

than-fourfold copy of a state for this entanglement makes the direct measurements possible, it simultaneously challenges the practical realization in experiment.

In this paper, we present a scheme to directly measure the genuine tripartite pure-state entanglement not only for qubit systems but also for  $(2 \otimes 2 \otimes n)$ -dimensional systems. The distinct advantage of this scheme is as follows. (1) *Only two simultaneous copies*: only a two-fold copy of the tripartite quantum state is required compared with the previous four-fold copy; (2) *a few projectors*:  $2n$  single-party plus two two-party rank-one projectors are much less than  $(16n^2 - 1)$  for the state tomography of a  $(2 \otimes 2 \otimes n)$ -dimensional state, which is the common advantage of all the related schemes, but only two two-party projectors are needed in contrast to six for three-qubit states in the previous scheme [23]; (3) *local operations*: the projective measurements are performed locally, which is the same as the previous schemes. As an example, we give a detailed demonstration and analysis on directly measuring the polarization entanglement of three photons in the frame of linear optics. This shows the feasibility of our scheme. In addition, the lower bounds of the genuine tripartite entanglement which is especially sufficient for detecting this type of entanglement are also provided for less adjustments of practical operations, but the cost, besides the lower bound, is that more projective measurement outcomes are needed. Finally, the influence of the imperfect experiment on our scheme is also discussed.

## II. THE GENUINE TRIPARTITE ENTANGLEMENT

Unlike bipartite entanglement, multipartite entanglement can be divided into many inequivalent entanglement classes. For example, three qubits can be entangled in two ways [25] and four qubits can be entangled in nine ways [26]. Therefore, usually a single scalar can only effectively characterize the entanglement of a single class or for some particular purposes. Even though multipartite entanglement of several quantum states has been well

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classified, three-tangle was first presented by Coffman, *et al.* [27] and is the most remarkable and widely accepted entanglement monotone for a general state (instead of the states of given class) to quantify the Greenberger-Horne-Zeilinger (GHZ) type entanglement of qubits. GHZ type entanglement describes genuine tripartite inseparability. It is distinguished from its opposite tripartite entanglement class (W type entanglement) that lies in the different robustness of the residual two-qubit entanglement against losing the third qubit. In fact, GHZ type entanglement can also be understood by the maximal extra average two-qubit entanglement induced by measurements on the third qubit with classical communication [28]. Furthermore, 3-tangle can be naturally generalized to a  $(2 \otimes 2 \otimes n)$ -dimensional quantum state in terms of concurrence and localizable concurrence. This can be explicitly given as follows. A tripartite quantum pure state of qubits defined in the  $(2 \otimes 2 \otimes n)$ -dimensional Hilbert space  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$  can be written in a computational basis as

$$|\psi\rangle_{ABC} = \sum_{i,j=0}^1 \sum_{k=0}^{n-1} a_{ijk} |i\rangle_A |j\rangle_B |k\rangle_C. \quad (1)$$

The reduced density matrix  $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|)$ . Considering the operations on qudit C, the maximal average concurrence of qubits A and B is characterized by the localizable concurrence [29, 30] which is given by

$$C_a(|\psi\rangle_{ABC}) = \sum_{i=1}^4 \lambda_i \quad (2)$$

and the minimal average concurrence is given by the concurrence [2] of  $\rho_{AB}$ , that is,

$$C(\rho_{AB}) = \max\{0, 2\lambda_1 - C_a(|\psi\rangle_{ABC})\}, \quad (3)$$

where  $\lambda_i$  denotes the square root of the eigenvalues of the matrix,

$$R = \rho_{AB}(\sigma_y \otimes \sigma_y) \rho_{AB}^* (\sigma_y \otimes \sigma_y) \quad (4)$$

in decreasing order with  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ . Thus the genuine tripartite entanglement monotone can be defined by [28]

$$\tau(|\psi\rangle_{ABC}) = \sqrt{C_a^2(|\psi\rangle_{ABC}) - C^2(\rho_{AB})}. \quad (5)$$

It is obvious that  $\tau^2(|\psi\rangle_{ABC})$  will become three-tangle for  $n = 2$ . Our first result shows that  $\tau(|\psi\rangle_{ABC})$  can be directly measured in experiment provided that only two-fold copy of  $|\psi\rangle_{ABC}$  is available. Our second result shows that the lower bound of  $\tau(|\psi\rangle_{ABC})$  can also be directly measured under the same condition with much simpler practical operations.

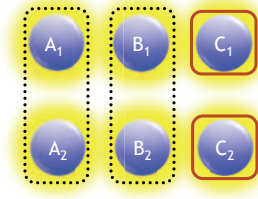


FIG. 1: (color online) Illustration of the scheme. The three balls in the yellow shades denote a copy of  $|\psi\rangle_{ABC}$  with the yellow shades representing the potential existence of quantum correlation. The dotted frame means a projective measurement  $P_-$  performed on the two particles inside and the solid frame denotes a single-qubit projective measurement  $P$ .

### III. THE MEASURABLE TRIPARTITE ENTANGLEMENT WITH TWO-FOLD COPY

It can be found that the matrix  $R$  given in Eq. (4) is the key to obtaining the genuine tripartite entanglement monotone  $\tau(|\psi\rangle_{ABC})$ . Next we will construct another measurable matrix that can extract all of the useful information related to  $\tau(|\psi\rangle_{ABC})$  from the matrix  $R$ .

Considering a set of orthonormal basis  $\{|a_k\rangle\}$  in  $\mathcal{H}_C$ ,  $|\psi\rangle_{ABC}$  can always be rewritten by  $|\psi\rangle_{ABC} = \sum_{k=0}^{n-1} |\varphi_k\rangle_{AB} |a_k\rangle_C$ , with  $|\varphi_k\rangle_{AB}$  denoting the bipartite pure state without normalization. Based on these  $|\varphi_k\rangle$ , one can easily construct the following symmetric matrix  $M$ :

$$M_{ij} = \langle \varphi_i^* |_{AB} (\sigma_y \otimes \sigma_y) | \varphi_j \rangle_{AB}. \quad (6)$$

Thus one can find that the following lemma holds.

**Lemma 1.-** *The set of the nonzero singular values of the matrix  $M$  is completely equal to the set of the square root of the eigenvalues of the matrix  $R$ .*

**Proof.** At first, we note that the reduced density matrix  $\rho_{AB}$  can be written as  $\rho_{AB} = \sum_{k=0}^{n-1} |\varphi_k\rangle_{AB} \langle \varphi_k|$ . So we can construct an  $n$ -dimensional matrix  $\Psi$  such that

$$\Psi = [|\varphi_0\rangle, |\varphi_1\rangle, |\varphi_2\rangle, \dots, |\varphi_{n-1}\rangle], \quad (7)$$

where we have omitted the subscripts  $(AB)$ . Thus it can be easily found that  $\rho_{AB} = \Psi \Psi^\dagger$ , and  $M$  can be rewritten as  $M = \Psi^T (\sigma_y \otimes \sigma_y) \Psi$ . In order to find the singular values, we will have to calculate the eigenvalues of  $MM^\dagger = \Psi^T (\sigma_y \otimes \sigma_y) \Psi \Psi^\dagger (\sigma_y \otimes \sigma_y) \Psi^*$ . It is obvious that  $MM^\dagger$  has the same eigenvalue set as the matrix  $\Psi \Psi^\dagger (\sigma_y \otimes \sigma_y) \Psi^* \Psi^T (\sigma_y \otimes \sigma_y) = \rho_{AB} (\sigma_y \otimes \sigma_y) \rho_{AB}^* (\sigma_y \otimes \sigma_y)$ , i.e., the matrix  $R$ . This finishes the proof. ■

On the basis of Lemma 1, one can draw the conclusion that the genuine tripartite entanglement  $\tau(|\psi\rangle_{ABC})$  will be completely determined once the matrix  $M$  is known. Considering the two-fold copy of  $|\psi\rangle_{ABC}$ , i.e.,  $|\psi\rangle_{A_1 B_1 C_1} \otimes |\psi\rangle_{A_2 B_2 C_2}$  in the Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2 = (\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{C_1}) \otimes (\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2} \otimes \mathcal{H}_{C_2})$ , one can define the projectors in the anti-symmetric subspace  $\mathcal{H}_{i_m} \wedge$

$\mathcal{H}'_{i_n}$  of  $\mathcal{H}_{i_m} \otimes \mathcal{H}'_{i_n}$  as

$$P_-^{(i_m i_n)} = |\Psi_{i_m i_n}^- \rangle \langle \Psi_{i_m i_n}^-| \quad (8)$$

with  $|\Psi_{i_m i_n}^- \rangle = \frac{1}{\sqrt{2}} (|0\rangle_{i_m} |1\rangle_{i_n} - |1\rangle_{i_m} |0\rangle_{i_n})$  written in the computational basis, where  $i = A, B$  corresponds to the subsystem A and B, and  $m, n = 1, 2$ , marks the different copies of  $|\psi\rangle_{ABC}$ . Thus we can arrive at another important lemma.

**Lemma 2.**-The entries of  $M$  can be given, subject to the two-fold copy of  $|\psi\rangle_{ABC}$ , by

$$\begin{aligned} M_{ij} &= 2 \left( \langle \Psi_{A_1 A_2}^- | \langle \Psi_{B_1 B_2}^- | \right) \left( |\varphi_i\rangle_{A_1 B_1} |\varphi_j\rangle_{A_2 B_2} \right) \quad (9) \\ &= 2 \left( \langle \Psi_{A_1 A_2}^- | \langle \Psi_{B_1 B_2}^- | \langle i_{C_1} | \langle j_{C_2} | \right) \left( |\psi\rangle_{A_1 B_1 C_1} |\psi\rangle_{A_2 B_2 C_2} \right), \quad (10) \end{aligned}$$

where  $|i_{C_1}\rangle$  and  $|i_{C_2}\rangle$  are the basis in  $\mathcal{H}_{C_1}$  and  $\mathcal{H}_{C_2}$ , respectively.

**Proof.** Expand  $\sigma_y \otimes \sigma_y$  in the computational basis, and we have  $\sigma_y^A \otimes \sigma_y^B = -|00\rangle_{AB} \langle 11| + |01\rangle_{AB} \langle 10| + |10\rangle_{AB} \langle 01| - |11\rangle_{AB} \langle 00|$ , where we use the indices A and B to mark different action objects. Therefore,  $M_{ij}$  can be rewritten as

$$M_{ij} = \langle \varphi_i^* |_{AB} (\sigma_y \otimes \sigma_y) |\varphi_j\rangle_{AB} \quad (11)$$

$$\begin{aligned} &= [-\langle 00|_{A_1 B_1} \langle 11|_{A_2 B_2} + \langle 01|_{A_1 B_1} \langle 10|_{A_2 B_2} \\ &+ \langle 10|_{A_1 B_1} \langle 01|_{A_2 B_2} - \langle 11|_{A_1 B_1} \langle 00|_{A_2 B_2}] |\varphi_i\rangle_{A_1 B_1} |\varphi_j\rangle_{A_2 B_2} \\ &= 2 \left( \langle \Psi_{A_1 A_2}^- | \langle \Psi_{B_1 B_2}^- | \right) \left( |\varphi_i\rangle_{A_1 B_1} |\varphi_j\rangle_{A_2 B_2} \right), \quad (12) \end{aligned}$$

which is exactly Eq. (9). Substitute  $|\varphi_i\rangle_{A_k B_k} = \langle i_{C_k} | \psi \rangle_{A_k B_k C_k}$  into Eq. (12) and one will easily obtain Eq. (10). ■

Lemma 2 expresses  $M_{ij}$  based on the inner product which is a probability amplitude, so it can not be directly related to the experiment. But a simple change will immediately arrive at the directly measurable quantities given in our following main theorem.

**Theorem 1.**-The absolute value  $|M_{ij}|$  can be directly measured by local projective measurements, i.e.,

$$|M_{ij}| = 2 \sqrt{\langle \psi |_{\varepsilon_1} \langle \psi |_{\varepsilon_2} \mathcal{A}_{ij} | \psi \rangle_{\varepsilon_1} | \psi \rangle_{\varepsilon_2}} \quad (13)$$

with  $\mathcal{A}$  a factorizable observable given by

$$\mathcal{A}_{ij} = P_-^{(A_1 A_2)} \otimes P_-^{(B_1 B_2)} \otimes P_i^{(C_1)} \otimes P_j^{(C_2)}, \quad (14)$$

where the subscripts  $\varepsilon_k$  denote the index  $A_k B_k C_k$  and  $P_m^{(C_k)} = |m\rangle_{C_k} \langle m|$  are the projectors subject to the basis  $|m\rangle$  in  $\mathcal{H}_{C_k}$ . Let  $M = U \Lambda U^T$ , with  $U$  unitary and  $\Lambda$  diagonal and positive. Then  $\tau(|\psi\rangle_{ABC})$  can be directly measured via  $|M_{ii}|$  with the optimal choice  $P_i^{(C_k)} = U^* |i\rangle_{C_k}$ .

**Proof.** Eq. (13) is a direct and obvious result of Lemma 2. This proof is omitted here. Next we will show that the genuine tripartite entanglement  $\tau(|\psi\rangle_{ABC})$  can

be obtained by the measurable  $|M_{ij}|$ . From Ref. [31], one can find that any operation  $Q$  operated on the qudit C can be equivalently described as  $\tilde{M} = Q^T M Q$ . Based on Takagi decomposition [32] of a complex symmetric matrix, one can always write  $M = U \Lambda U^T$  where  $U$  is a unitary matrix and  $\Lambda$  is a diagonal matrix with the diagonal entries corresponding to the singular values of  $M$ . In this sense, one can always select a proper local unitary operation  $Q$  on qudit C such that  $Q^T U = I$  which corresponds to  $\tilde{P}_i^{(C_1)} = U^* |i\rangle_{C_1}$  and  $\tilde{P}_j^{(C_2)} = U^* |j\rangle_{C_2}$ .

With such a choice, one will obtain that  $\tilde{M} = |\tilde{M}| = \Lambda$ . In other words, so long as we choose the optimal projective measurements on  $C_1$  and  $C_2$ ,  $\tilde{M}_{ii}$  just corresponds to  $\lambda_i$ , i.e., the singular values of  $M$ . This means that  $\tau(|\psi\rangle_{ABC})$  can be measured directly and locally based on Eq. (5). ■

The above proof implies very important contents. One can find that  $\tilde{M}_{ij}$ ,  $i \neq j$ , vanish once the optimal projective measurements on  $C_k$  are achieved. It means that, if the optimal projector on  $C_1$  and  $C_2$  are different, there will not be any output corresponding to the projective measurements  $P_-^{(A_1 A_2)}$  and  $P_-^{(B_1 B_2)}$ . So this becomes an important index by which one can signal when the optimal measurement basis has been achieved in the practical adjusting procedure.

An intuitive illustration of this scheme is sketched in Fig. 1. Suppose we have a pair of entangled tripartite pure states  $|\psi\rangle_{A_1 B_1 C_1}$  and  $|\psi\rangle_{A_2 B_2 C_2}$ . The projective measurements  $P_i^{(C_1)}$  and  $P_j^{(C_2)}$  are performed on qubits  $C_1$  and  $C_2$ , respectively. At the same time, joint projective measurement  $P_-^{(A_1 A_2)}$  is performed on  $A_1, A_2$  and  $P_-^{(B_1 B_2)}$  is performed on  $B_1, B_2$ . Adjust the measurement basis of  $P_i^{(C_1)}$  and  $P_j^{(C_2)}$  such that no signal is output from the measurement terminals  $P_-^{(A_1 A_2)}$  and  $P_-^{(B_1 B_2)}$  when  $i \neq j$ . At this moment,  $\tilde{M}_{ii}$  can be expressed by

$$\tilde{M}_{ii} = p_i^{(C_1)} p_i^{(C_2)} p_-^{(A_1 A_2)} p_-^{(B_1 B_2)} \quad (15)$$

with  $p_k^{(\cdot)}$  denoting the probability corresponding to the projective measurements on  $(\cdot)$ . So  $\tau(|\psi\rangle_{ABC})$  can be easily obtained.

#### IV. MEASURING 3-TANGLE OF QUBITS IN LINEAR OPTICAL EXPERIMENT

We take the linear optical experiment of our scheme as an example. The experimental setup is briefly sketched in Fig. 2. Two entanglement resources are used to generate three entangled polarized photons with the state  $|\psi\rangle_{A_1 B_1 C_1}$  and  $|\psi\rangle_{A_2 B_2 C_2}$ , respectively. As a demonstration, one can use the same setups as Ref. [33] to produce the polarized GHZ state of three photons. Each group of entangled photons are distributed into three paths represented by the labels of the corresponding qubits, respectively. Let photons  $A_1$  and  $A_2$  go through a beam splitter

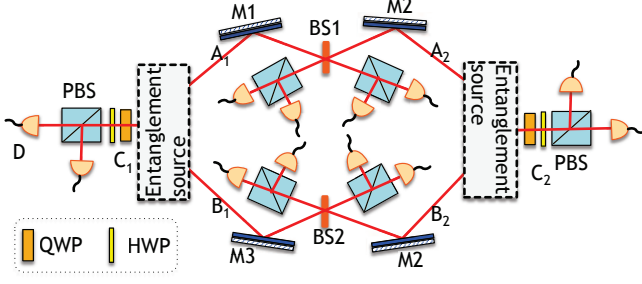


FIG. 2: (Color online) Brief diagram for the linear optical realization. Three photons are distributed to three paths. The BS+PBS is used to realize the projective measurement  $P_{-}^{(\cdot)}$  and the HWP+QWP is used to implement the projective measurement  $P_{C_k}$  on any possible basis. The entanglement sources can be replaced by the experiment in Ref. [33] for a simple demonstration of the GHZ state.

(BS), then undergo a polarized beam splitter (PBS) and finally be detected by single photon detectors. At the same time, let photons  $B_1$  and  $B_2$  go through another set of similar setups as  $A_1$  and  $A_2$ . But we let photons  $C_1$  and  $C_2$  first go through a quarter wave plate (QWP) and a half wave plate (HWP) and then arrive at a PBS. One can practically adjust the QWP and HWP to look for the optimal basis for projective measurements  $P_i^{(C_1)}$  and  $P_j^{(C_2)}$  which can be achieved until no effective click [34] corresponding to  $P_{-}^{(A_1 A_2)}$  and  $P_{-}^{(B_1 B_2)}$  is recorded. The final measurement statistics can be achieved by recording the clicks of each detector. It is worthwhile to note that such an adjustment of basis has been employed in a previous experiment [35], and hence it is not at all necessary to worry about the feasibility of the choice of the optimal basis.

## V. THE MEASURABLE LOWER BOUND

Even though the adjustment of projective measurements on qubit  $C_k$  is practically feasible, one could not be satisfied with this. Next, we present a weak scheme that does not need the optimal  $P_{-}^{(A_1 A_2)}$  and  $P_{-}^{(B_1 B_2)}$ . So this will greatly simplify the practical operations, but the cost is that one needs more measurement outcomes and only a lower bound of  $\tau(|\psi\rangle_{ABC})$  is obtained, even though the bounds are very good.

From Eq. (5), one can easily find that

$$\begin{aligned} \tau(|\psi\rangle_{ABC}) &= \begin{cases} \sum_{i=1}^4 \lambda_i, & \lambda_1 \leq \sum_{i=2}^4 \lambda_i \\ 2\sqrt{\lambda_1 \sum_{i=2}^4 \lambda_i}, & \lambda_1 > \sum_{i=2}^4 \lambda_i \end{cases} \\ &\geq 2\sqrt{\sum_{i=1}^4 \lambda_i^2 - \lambda_1^2}. \end{aligned} \quad (16)$$

This is a good lower bound in that it is a sufficient and necessary condition for GHZ type inseparability of  $|\psi\rangle_{ABC}$ . It can be seen from that the lower bound vanishes if and only if the matrix  $M$  is rank-one which is equivalent to  $\tau$ . Based on the upper bound of the singular value of a matrix [32], one can find that  $\lambda_1$  can be well bounded by

$$\begin{aligned} \lambda_1 &\leq \sigma_U(q) \\ &= \left[ \max_k \sum_{j=1}^n |M_{jk}|^{2q} \right]^{1/2} \left[ \max_j \sum_{k=1}^n |M_{jk}|^{2(1-q)} \right]^{1/2} \end{aligned} \quad (17)$$

for  $q \in [0, 1]$ . Thus we have

$$\tau(|\psi\rangle_{ABC}) \geq 2\sqrt{\text{Tr}MM^\dagger - \min_{q \in [0,1]} \sigma_U^2(q)}. \quad (18)$$

Some simple bounds can be found when  $q = 0, \frac{1}{2}, 1$ .

It is obvious that  $\text{Tr}MM^\dagger = \text{Tr}\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y) = \text{Tr}[(\rho_{A_1 B_1} \otimes \rho_{A_2 B_2})P_{-}^{(A_1 A_2)} \otimes P_{-}^{(B_1 B_2)}]$  which shows that  $\text{Tr}MM^\dagger$  can be directly measured by local projective measurements with a two-fold copy of the state. In addition,  $\sigma_U(q)$  given in Eq. (17) is described by  $|M_{jk}|$ , which can be obtained by the measurement statistics produced by  $n(n+1)/2$  measurements including  $P_i^{(C_1)}$  and  $P_j^{(C_2)}$ . From the lower bound point of view, it is not necessary to adjust the basis for  $P_i^{(C_1)}$  and  $P_j^{(C_2)}$ . But the optimal choice of these two projectors can greatly improve the lower bound. So the lower bound is locally measurable provided that two copies of the states are available.

In fact, for three qubits, an alternative lower bound that could be relatively tight can be given by

$$\begin{aligned} \tau_{\text{qubit}}(|\psi\rangle_{ABC}) &= 2\sqrt{|\det M|} \\ &\geq 2\sqrt{||M_{00}||M_{11}| - |M_{01}||M_{10}|} \end{aligned} \quad (19)$$

This bound can be easily proved by Eq. (6) for  $i, j \leq 1$ . So Eq. (5) can be directly related to the determinant of matrix  $M$ . It is obvious that all of the elements in the lower bound can be directly measured based on the above scheme, so the lower bound can be experimentally determined.

## VI. DISCUSSIONS AND CONCLUSION

No experiment is perfect, so we have to know to what degree the measurement results are acceptable. In this scheme, in order to reduce the copies of the measured state, a key point is to adjust the projective measurements on  $C_k$  such that no effective outputs corresponding to  $P_{-}^{(A_1 A_2)} \otimes P_{-}^{(B_1 B_2)}$  are generated. However, there could be a small probability  $\epsilon$  to detect photons in a practical

scenario. Thus, the practical  $|M_{ii}|$  can always be formally given by  $|M_{ii}| = \lambda_i + \epsilon\Delta$  in first order. This will lead to a small ( $\sim \epsilon$ ) deviation for the exact tripartite entanglement, but a little smaller lower bound for Eq. (19). In addition, the previous similar jobs tried to compensate for the experimental imperfection, which could imply that more prior information should be known. Here instead of doing this, we will mainly find out the potential errors. Without loss of generality, we only suppose that the prepared state is a quasi-pure state, that is,

$$\rho_{A_k B_k C_k} = (1 - \epsilon_k) |\psi\rangle \langle \psi|_{A_k B_k C_k} + \epsilon_k \varrho_k, \quad (20)$$

where  $\epsilon_k \ll 1$  and  $\varrho_k$  is a general tripartite density matrix with the subscript  $k = 1, 2$  distinguishing different copies. Since the copy of the state is generated by another setup, it is reasonable to assume that

$$|\psi\rangle_{A_2 B_2 C_2} = \sqrt{1 - \epsilon_0^2} |\psi\rangle_{A_1 B_1 C_1} + \epsilon_0 |\phi\rangle \quad (21)$$

with  $\epsilon_0 \ll 1$  and  $\langle \psi | \phi \rangle = 0$ . Substitute Eqs. (20) and (21) into Eq. (14), and one will find that  $|M'_{ij}|$ , corresponding to the imperfect preparation and copy, can be written in first order of  $\epsilon_i$  as

$$|M'_{ij}| \sim \sqrt{(1 - \epsilon_1 - \epsilon_2) |M_{ij}|^2 + \tilde{\epsilon} N},$$

where  $\tilde{\epsilon} = \max\{\epsilon_0, \epsilon_1, \epsilon_2\}$  and  $N$  is not explicitly given here. All of the above analysis shows that the imper-

fect experiment will lead to a small deviation (about  $\max\{\epsilon, \tilde{\epsilon}\}$ ) of the real value. However, if the entanglement of  $|\psi\rangle$  is so small that  $\tau \sim \epsilon$ , that is, noise drowns out the signal, then this scheme cannot detect any entanglement which is similar to all the relevant jobs.

In summary, we have found that the high-dimensional tripartite entanglement can be locally measured with only a two-fold copy of the state. For simplicity, we also provide a good lower bound for entanglement which will simplify the practical operations but require more measurement outcomes. As a demonstration, we consider how to measure the three-tangle for three entangled qubits based on linear optical setups. The current scheme is only fit for a pure state which may not be so practical. However, needless to say, for the measurable entanglement with less copies of the state, even a simple lower bound for tripartite entanglement is not available in entanglement theory. We think this scheme could be an important step towards the more general cases.

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